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# The contribution of thermal electron–positron pairs to the thermodynamic properties of black-body radiation

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**Abstract.** The contribution of thermal electron–positron pairs to the thermodynamic properties of black-body radiation (BBR) is considered. This contribution is examined in the vicinity of the temperature corresponding to the electron rest mass energy $T_c = m_e c^2/k$, in which it becomes appreciable. The correction factors are defined as the ratio of extended thermodynamic expressions of BBR (which also include the contributions from thermal pairs) to the familiar expressions of it. Then the variation of these factors with temperature is given. It is shown that while they have the same value for each property in both high and low temperature regimes ($T \gg T_c$ and $T \ll T_c$), they have different values about 0.5 $T_c$. It is found that for BBR, the ratio of the energy density to the pressure has a maximum about 0.33 $T_c$ and also the ratio of the specific heats has a minimum about 0.4 $T_c$.

## 1. Introduction

At ordinary temperatures, the contribution from black-body radiation (BBR) to the thermodynamic properties of a system can be neglected. On the other hand, if the temperature of the system is high enough, this contribution becomes dominant. Therefore, thermodynamic properties of BBR are quite important in high temperature systems.

If the temperature of a system is much lower than the temperature corresponding to the electron rest mass energy ($T \ll T_c = m_e c^2/k$), it can be assumed that BBR consists of photons only. In this case, familiar thermodynamic expressions of BBR are valid. At temperatures near $T_c$ ($T_c = 5.938 \times 10^9$ K), however, BBR contains a considerable number of thermal electron–positron pairs in addition to photons. The number of pairs increases rapidly with increasing temperature and consequently thermal pairs make an important contribution to the thermodynamic properties of BBR beginning from a temperature about $0.1T_c$. The expressions of BBR should be extended by considering the contribution of thermal pairs so that they are also valid at these temperatures.

In the literature, this problem has been investigated for high and low temperature regimes ($T \gg T_c$ and $T \ll T_c$) [1–5]. However, the variation of thermodynamic properties of BBR at temperatures about $T_c$ ($0.1T_c < T < 10T_c$) has not been studied previously. Here this variation is examined. The outline of this paper is as follows. In section 2, thermodynamic properties of thermal electron and positron gases are determined analytically without any temperature restrictions such as $T \ll T_c$ or $T \gg T_c$. In section 3, correction factors are defined as the ratio of extended expressions of BBR to the familiar expressions of it. In section 4, the variation of both these factors and some thermodynamic ratios with the temperature are examined about $T_c$. It is seen that while the correction factors have the same value for each property at both $T \ll T_c$ and $T \gg T_c$, their values are different about

$0.5T_c$. *It is found that the ratio of the energy density to the pressure has a maximum about* $0.33T_c$ and the ratio of the specific heats has a minimum about $0.4T_c$. Assumptions and restrictions made in the calculations are as follows.

(i) The non-thermal electron density ($n^0_{e^-}$) in the system is restricted as $n^0_{e^-} < 10^{30}$ m$^{-3}$. So, it can be assumed that the electrons and positrons are of equal density ($n_{e^-} = n_{e^+}$) for the temperatures at which the contribution of thermal pairs is appreciable. Since $10^{30}$ m$^{-3}$ is approximately equal to the total (not only free) electron density of the lead, this assumption is valid in a large density scale.

(ii) Thermal electrons and positrons interact with both each other and photons. These interactions cause additional terms in the expressions obtained under the assumption that there is no interaction. For $T \ll T_c$ and $T \gg T_c$, the ratios of the additional terms to the expressions for the non-interacting case are given in the literature [1, 3]. In the case of $T \gg T_c$, this ratio for the free energy of BBR goes to its maximum absolute value which is approximately 1/378. Since this value is negligibly small, interactions here are neglected.

(iii) If $T \approx 207T_c$, there are not only $e^-$–$e^+$ pairs but also many muon–anti-muon pairs in the system. At higher temperatures, the heavier thermal particle–anti-particle pairs are created. If the interactions are negligible, the expressions for $e^-$–$e^+$ pairs can also be used for new thermal pairs by using the related $T_c$ values. To avoid repetition, only $e^-$–$e^+$ pairs are considered in this work and so the temperature is restricted as $T < 207T_c$. However, it should be emphasized that when the temperature is high enough for the creation of hadron pairs, these expressions are insufficient due to the existence of strong interaction among them.

## 2. Thermodynamic properties of thermal electron and positron gases

Since it is assumed that $n_{e^-} = n_{e^+}$ about $T_c$, the chemical potentials of $e^-$ and $e^+$ are equal to zero for thermodynamic equilibrium ($\mu_{e^-} = \mu_{e^+} = 0$). Thus, using the relativistic energy relation, electron and positron densities can be written as follows [5]

$$n_{e^-}(T) = n_{e^+}(T) = \frac{8\pi}{(ch)^3}\int_{m_e c^2}^{\infty} \frac{\varepsilon(\varepsilon^2 - m_e^2 c^4)^{1/2}}{1+\exp(\varepsilon/kT)}\mathrm{d}\varepsilon. \tag{1}$$

If $\chi = \varepsilon/kT$ is used as a variable of integration, equation (1) becomes

$$n_{e^-}(\alpha) = n_{e^+}(\alpha) = \frac{T_c^3}{\Lambda}\frac{1}{\alpha^3}\int_{\alpha}^{\infty} \frac{\chi(\chi^2-\alpha^2)^{1/2}}{1+\exp(\chi)}\mathrm{d}\chi \tag{2}$$

where $\Lambda = \pi^4 kc/(60\sigma_0)$, $\sigma_0$ is the Stefan–Boltzmann constant and $\alpha = \alpha(T) = T_c/T$. Considering $\exp(\chi) > 1$, if $1/[1+\exp(\chi)]$ is expanded as a series in power of $\exp(\chi)$ then equation (2) is

$$n_{e^-}(\alpha) = n_{e^+}(\alpha) = \frac{T_c^3}{\Lambda}\frac{1}{\alpha^3}\int_{\alpha}^{\infty} \chi(\chi^2-\alpha^2)^{1/2}\sum_{j=1}^{\infty}(-1)^{j+1}\exp(-j\chi)\mathrm{d}\chi. \tag{3}$$

Equation (3) can be solved analytically [6]. Thus the thermal electron and positron densities are

$$n_{e^-}(\alpha) = n_{e^+}(\alpha) = \frac{T_c^3}{\Lambda}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{K_2(-j\alpha)}{j\alpha} \tag{4}$$

where $K_2(j\alpha)$ is the modified (or hyperbolic) Bessel function of second order.

Similarly, free energy expressions for $e^-$ and $e^+$ gases are obtained by using the general description of it for a fermion gas [5] as follows

$$F_{e^-}(\alpha) = F_{e^+}(\alpha) = -\frac{T_c^4 k}{3\Lambda}\frac{V}{\alpha^4}\int_\alpha^\infty \frac{(\chi^2-\alpha^2)^{3/2}}{1+\exp(\chi)}\mathrm{d}\chi = -\frac{T_c^4 k}{\Lambda} V \sum_{j=1}^{\infty}(-1)^{j+1}\frac{K_2(j\alpha)}{(j\alpha)^2}. \tag{5a, b}$$

Using the definitions of energy, pressure and entropy in terms of the free energy, they can be expressed in the form

$$E_{e^-}(\alpha) = E_{e^+}(\alpha) = \frac{T_c^4 k}{3\Lambda}\frac{V}{\alpha^4}\int_\alpha^\infty \frac{(\chi^2-\alpha^2)^{3/2}[\chi - \exp(-\chi) - 1]}{2+\exp(\chi)+\exp(-\chi)}\mathrm{d}\chi \tag{6a}$$

$$= \frac{T_c^4 k}{\Lambda} V \sum_{j=1}^{\infty}(-1)^{j+1}\left(\frac{K_2(j\alpha)}{(j\alpha)^2} + \frac{K_1(j\alpha)+K_3(j\alpha)}{2j\alpha}\right) \tag{6b}$$

$$P_{e^-}(\alpha) = P_{e^+}(\alpha) = \frac{T_c^4 k}{3\Lambda}\frac{1}{\alpha^4}\int_\alpha^\infty \frac{(\chi^2-\alpha^2)^{3/2}}{1+\exp(\chi)}\mathrm{d}\chi = \frac{T_c^4 k}{\Lambda}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{K_2(j\alpha)}{(j\alpha)^2} \tag{7a, b}$$

$$S_{e^-}(\alpha) = S_{e^+}(\alpha) = \frac{T_c^3 k}{3\Lambda}\frac{V}{\alpha^3}\int_\alpha^\infty \frac{\chi(\chi^2-\alpha^2)^{3/2}}{2+\exp(-\chi)+\exp(\chi)}\mathrm{d}\chi \tag{8a}$$

$$= \frac{T_c^3 k}{\Lambda} V \sum_{j=1}^{\infty}(-1)^{j+1}\left(\frac{2K_2(j\alpha)}{j^2\alpha} + \frac{K_1(j\alpha)+K_3(j\alpha)}{2j}\right). \tag{8b}$$

Similar expressions to equations (4), (6*b*) and (7*b*) are also given in [2]. By using equations (6*a*) and (6*b*), the specific heat at constant volume is obtained as

$$C_V^{e^-}(\alpha) = C_V^{e^+}(\alpha) = -\frac{\alpha^2}{T_c}\left(\frac{\partial E}{\partial \alpha}\right)_V = \frac{T_c^4 k}{3\Lambda}\frac{V}{\alpha^3}\int_\alpha^\infty \frac{[3\alpha^2(\chi^2-\alpha^2)^{1/2} + 4(\chi^2-\alpha^2)^{3/2}]}{2+\exp(\chi)+\exp(-\chi)} \times[\chi - \exp(-\chi) - 1]\mathrm{d}\chi \tag{9a}$$

$$= \frac{T_c^3 k}{4\Lambda} V \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}[6K_3(j\alpha) + 3j\alpha K_2(j\alpha) + j\alpha K_4(j\alpha)]. \tag{9b}$$

If the temperature and volume are chosen as independent variables, the specific heat at constant pressure is

$$C_P^i(\alpha) = C_V^i(\alpha) + [P_i(\alpha) + E_i(\alpha)/V]\left(-\frac{\alpha^2}{T_c}\right)\left(\frac{\partial V}{\partial \alpha}\right)_{P_i} \qquad i = e^-, e^+. \tag{10}$$

By making the substitution $r_i = r_i(\alpha) = E_i(\alpha)/[VP_i(\alpha)]$, to eliminate the derivative at constant pressure, equation (10) becomes

$$C_P^i(\alpha) = \left(2 + \frac{1}{r_i(\alpha)}\right)C_V^i(\alpha) - \frac{P_i(\alpha)V}{T_c}\alpha^2\left(1 + \frac{1}{r_i(\alpha)}\right)\frac{\mathrm{d}r_i(\alpha)}{\mathrm{d}\alpha}. \tag{11}$$

Thus $C_P^i(\alpha)$ can be determined from equations (6*a*, *b*), (7*a*, *b*) and (9*a*, *b*).

In the case of $\alpha \gg 1$ ($T_c \gg T$), high-order terms in the series solutions become negligible and the term $\exp(\chi)$ in the denominator of the integrands becomes dominant. In this temperature regime, number densities of the thermal electrons and positrons are very small so that Boltzmann statistics can be used instead of Fermi statistics. Thus the expressions converge to those of the relativistic ideal Boltzmann gas [2, 7]. The only difference between them is that chemical potential is zero in the expressions here. On the other hand, in the case of $\alpha \ll 1$ ($T_c \ll T$), rest mass energies of the thermal pairs can be neglected because they are small compared with their kinetic energies. Therefore they converge to expressions of extra relativistic Fermi gas [5, 8]. This can be seen by using limiting properties of modified Bessel functions [9] in the series solutions or by taking $\alpha = 0$ in the integrals.

*Series solutions converge very rapidly due to the properties of modified Bessel functions.* Therefore they can be calculated with less computing time in comparison with the integral expressions.

## 3. Determination of the correction factors for thermodynamic properties of BBR

Theoretically, there are always thermal electron and positron constituents of BBR at every temperature. Because the interactions between them are negligibly small, the thermodynamic properties of BBR are simply written as a summation of the properties of its constituents. Therefore, any thermodynamic property of BBR $Q(\alpha)$ can be written as

$$\begin{aligned} Q(\alpha) &= Q_\gamma(\alpha) + Q_{e^-}(\alpha) + Q_{e^+}(\alpha) = Q_\gamma(\alpha) + 2Q_{e^-} Q_\gamma(\alpha)\left(1 + \frac{2Q_{e^-}(\alpha)}{Q_\gamma(\alpha)}\right) \\ &= Q_\gamma(\alpha) CF_Q(\alpha) \end{aligned} \tag{12}$$

where $Q_\gamma(\alpha)$, $Q_{e^-}(\alpha)$ and $Q_{e^+}(\alpha)$ represent any thermodynamic property of thermal photon, electron and positron gases respectively. Here $CF_Q(\alpha)$ may be called the correction factor for a $Q$ property. The thermodynamic properties of a thermal photon gas (familiar BBR) are given in terms of $\alpha$ and some constants described here as follows.

$$N_\gamma(\alpha) = \frac{2T_c^3}{\Lambda}\zeta(3)\frac{V}{\alpha^3} \tag{13}$$

$$E_\gamma(\alpha) = \frac{6T_c^4}{\Lambda}k\zeta(4)\frac{V}{\alpha^4} \tag{14}$$

$$P_\gamma(\alpha) = \frac{2T_c^4}{\Lambda}k\zeta(4)\frac{1}{\alpha^4} \tag{15}$$

$$S_\gamma(\alpha) = \frac{8T_c^3}{\Lambda}k\zeta(4)\frac{V}{\alpha^3} \tag{16}$$

$$C_V^\gamma(\alpha) = \frac{24T_c^3}{\Lambda}k\zeta(4)\frac{V}{\alpha^3} \tag{17}$$

$$C_P^\gamma(\alpha) = \frac{56T_c^3}{\Lambda}k\zeta(4)\frac{V}{\alpha^3} \tag{18}$$

where $\zeta(3)$ and $\zeta(4)$ are Riemann zeta functions [5]. Correction factors can be obtained by using equations (4), (6)–(12) and (13)–(18). Thus the correction factor for thermal particle density is

$$CF_N(\alpha) = 1 + \frac{\alpha^2}{\zeta(3)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}K_2(j\alpha) \tag{19}$$

those for energy, pressure and entropy are

$$CF_E(\alpha) = 1 + \frac{\alpha^3}{3\zeta(4)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\left(\frac{K_2(j\alpha)}{j\alpha} + \frac{K_1(j\alpha) + K_3(j\alpha)}{2}\right) \tag{20}$$

$$CF_P(\alpha) = 1 + \frac{\alpha^2}{\zeta(4)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j^2}K_2(j\alpha) \tag{21}$$

$$CF_S(\alpha) = 1 + \frac{\alpha^3}{4\zeta(4)} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \left( \frac{2K_2(j\alpha)}{j\alpha} + \frac{K_1(j\alpha) + K_3(j\alpha)}{2} \right) \quad (22)$$

that for the specific heat at constant volume is

$$CF_{C_V}(\alpha) = 1 + \frac{\alpha^3}{48\zeta(4)} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} [6K_3(j\alpha) + 3j\alpha K_2(j\alpha) + j\alpha K_4(j\alpha)] \quad (23)$$

and that for the specific heat at constant pressure is

$$CF_{C_P}(\alpha) = 1 + \frac{\Lambda\alpha^3}{28\zeta(4) T_c^3 k V} C_P^{e^-}(\alpha). \quad (24)$$

## 4. Results and comments

The variation of the correction factors as a function of $T$ becomes considerable in the vicinity of $T_c$ ($T_c = 5.938 \times 10^9$ K). Therefore, $10^8 < T\,(\mathrm{K}) < 10^{11}$ is considered. Variation of $CF_N(T)$ in this temperature interval is given in figure 1. In the high temperature regime ($T \gg T_c$), it is seen that the correction factor goes to 2.5 ($1 + 2 \times \frac{3}{4} = 2.5$) obtained by using the well known $\frac{3}{4}$ value [5, 8].

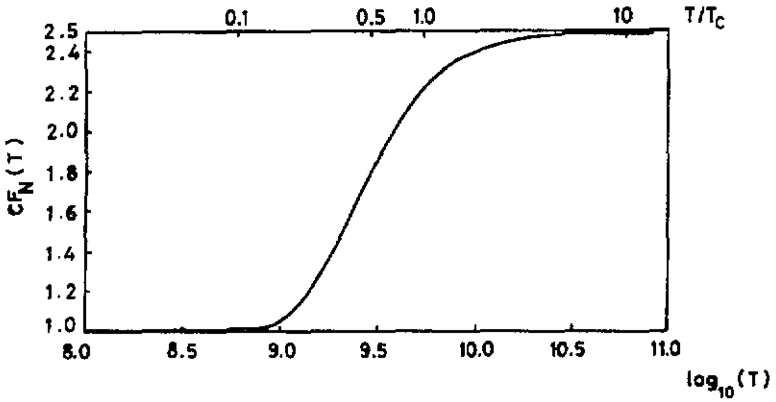


**Figure 1.** Variation of the correction factor for the number density of thermal particle $CF_N(T)$ plotted against both $\log_{10}(T)$ and $T/T_c$.

For energy, pressure and entropy of BBR, the variations of the correction factors depending on the temperature are also given in figure 2. At $T \gg T_c$, all of them go to 2.75 ($1 + 2 \times \frac{7}{8} = 2.75$) obtained by using the well known $\frac{7}{8}$ value [5, 8]. On the other hand, for $T \ll T_c$ the contribution of thermal pairs is negligible because their number is small compared with the photon number, so the correction factors are equal to unity. However, it is seen that $CF_E(T) > CF_S(T) > CF_P(T)$ at temperatures near $0.5T_c$. This situation can be explained as follows. While the pressure of BBR is only related to the kinetic energy density, its energy also contains the rest mass energies of particles. Therefore the correction to the energy is always greater than that to the pressure. Similarly, the entropy of BBR is related to both the number densities of thermal particles and their kinetic energy. Thus the correction to the entropy is also greater than that to the pressure. In the case of $T \gg T_c$, since the rest mass energies of particles are negligible compared with their kinetic energies, they behave like particles without mass. So the corrections converge to the same

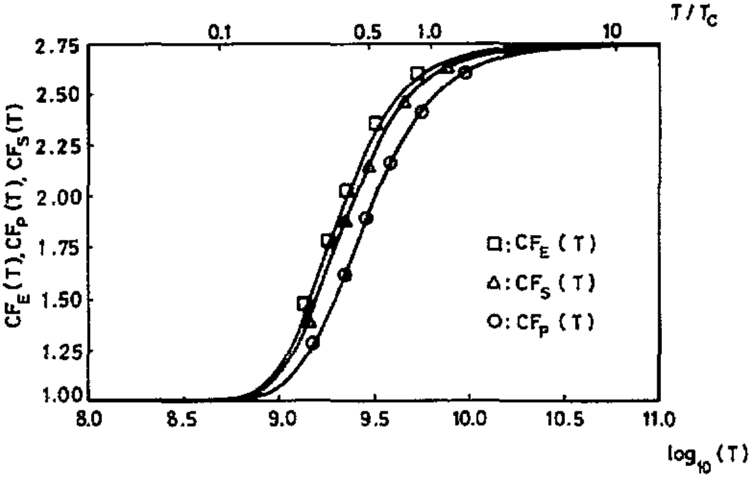


**Figure 2.** Variation of the correction factors for energy, pressure and entropy $CF_E(T)$, $CF_P(T)$, $CF_S(T)$ plotted against both $\log_{10}(T)$ and $T/T_c$.

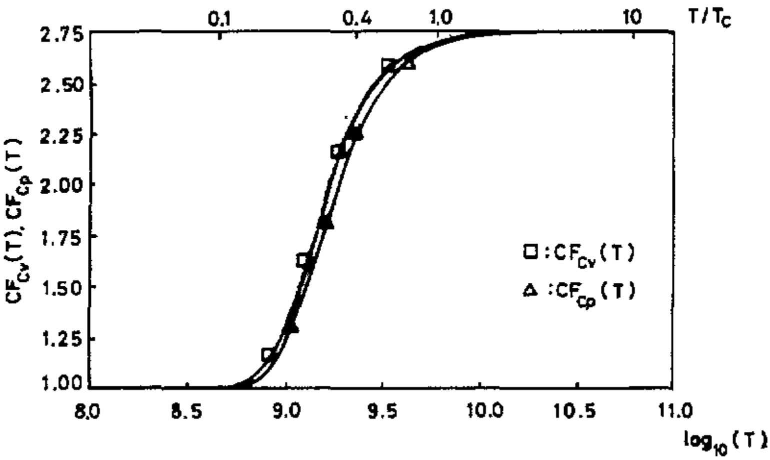


**Figure 3.** Variation of the correction factors for specific heats at constant volume and pressure $CF_{C_V}(T)$, $CF_{C_P}(T)$ plotted against both $\log_{10}(T)$ and $T/T_c$.

value. $CF_{C_V}(T)$ and $CF_{C_P}(T)$ have the forms of curves in figure 3. In the vicinity of $0.4T_c$, $CF_{C_V}(T) > CF_{C_P}(T)$ because $CF_E(T) > CF_P(T)$.

The specific heats ratio of the BBR can be written in the form $C_P(T)/C_V(T) = \frac{7}{3}CF_{C_P}(T)/CF_{C_V}(T)$. In figure 4, it is seen that this ratio has a minimum about $0.4T_c$ while it is equal to $\frac{7}{3}$ at both $T \ll T_c$ and $T \gg T_c$. This minimum results from the difference between $CF_{C_V}(T)$ and $CF_{C_P}(T)$. At this point, it should be emphasized that the ratio of the specific heats is different from the isentropic exponent since the specific heats here are relativistic.

For BBR, the ratio of the energy density to the pressure can be expressed as $E(T)/[P(T)V] = 3CF_E(T)/CF_P(T)$. Figure 5 shows how this ratio depends on $T$. There is a maximum about $0.33T_c$ because the $CF_E(T)$ and $CF_P(T)$ are different. This maximum may be important, especially in some astrophysical and cosmological phenomena,

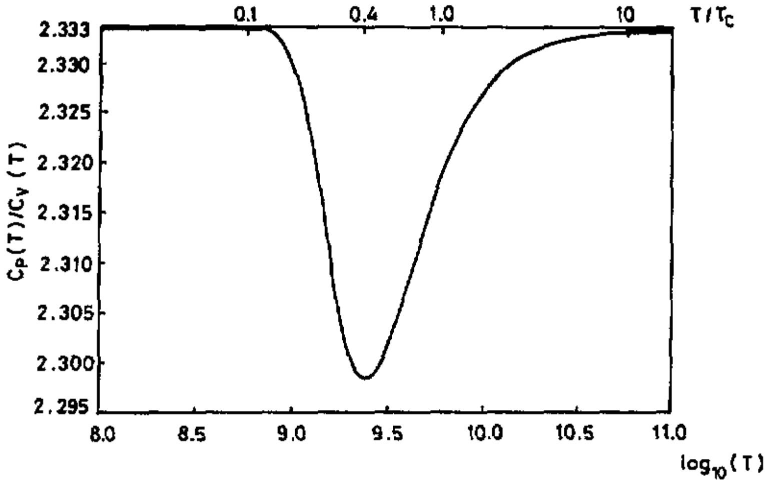


**Figure 4.** Ratio of the specific heats of BBR plotted against both $\log_{10}(T)$ and $T/T_c$.

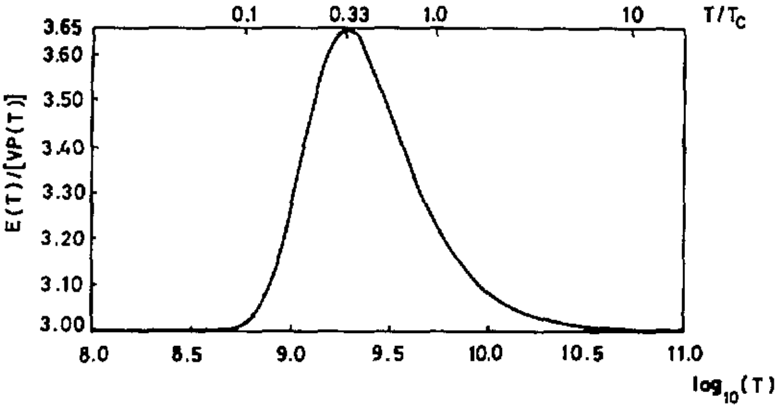


**Figure 5.** Ratio of the energy density to the pressure of BBR plotted against both $\log_{10}(T)$ and $T/T_c$.

since the pressure balances the gravitational attraction. As a result of this, it is expected that gravitational collapse of a system becomes faster when $T$ goes to $0.33T_c$.

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